Precalculus Workshop - Algebra

Sets of Numbers

The sets of numbers you may work with in your calculus course are:

1. the natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$
2. the integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
3. the rational numbers $\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$
4. the real numbers $\mathbb{R}$, which is the union of the rational numbers and the irrational numbers. An irrational number is any real number that is not rational (for example, π or $\sqrt{2}$.)

Fraction Arithmetic

1. Addition/Subtraction: find a common denominator, then add the numerators. Always use the lowest common denominator.
2. Multiplication: cancel common factors, then multiply the numerators and multiply the denominators.
3. Division: multiply the first fraction by the reciprocal of the second.

VIDEO EXAMPLE 1.1 - COMPLEX FRACTIONS

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Basic Exponent Laws

Let a, b, m and n be any real numbers. Then the basic exponent rules are

1. $a^0 = 1$ $(a \neq 0)$
2. $a^{-n} = \dfrac{1}{a^n}$ $(a \neq 0)$
3. $a^m a^n = a^{m+n}$
4. $\dfrac{a^m}{a^n} = a^{m-n}$ $(a \neq 0)$
5. $(a^m)^n = a^{mn}$

VIDEO EXAMPLE 1.2 - BASIC EXPONENT RULES

Distributive Laws and Rational Exponents

For a real number a and a positive integer n, we use the nth-root notation. If m is also an integer, then we have the following:

1. $a^{1/n} = \sqrt[n]{a}$ (Note: these expressions are not defined when n is even and $a < 0$.)
2. $a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$.

Let a, b, m and n be any real numbers. The distributive laws of exponents are

1. $(ab)^m = a^m b^m$
2. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ $(b \neq 0)$.

If we view rational exponents using nth-root notation, then we have the following as a consequence of the previous two properties:

1. $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$ or $(ab)^{1/n} = a^{1/n}b^{1/n}$
2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ or $\left(\dfrac{a}{b}\right)^{1/n} = \dfrac{a^{1/n}}{b^{1/n}}$ $(b \neq 0)$.

VIDEO EXAMPLE 1.3 - DISTRIBUTIVE LAWS FOR EXPONENTS

VIDEO EXAMPLE 1.4 - SIMPLIFYING RATIONAL EXPONENTS 1

VIDEO EXAMPLE 1.5 - SIMPLIFYING RATIONAL EXPONENTS 2

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Polynomials

A polynomial of degree n is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

where n is a positive integer and $a_n \neq 0$. The constants a_0, a_1, ..., a_n are called coefficients. When the degree of $P(x)$ is 2, the polynomial function is called a quadratic function. When the degree of $P(x)$ is 1, the polynomial function is called a linear function.

Polynomial Arithmetic

When working with polynomial expressions, we can use the following:

1. Addition/Subtraction: add/subtract coefficients of equal powers
2. Multiplication: use the distributive property
3. Division: use the long division algorithm, or synthetic division.

VIDEO EXAMPLE 1.6 - SIMPLIFYING POLYNOMIAL EXPRESSIONS

VIDEO EXAMPLE 1.7 - LONG DIVISION AND SYNTHETIC DIVISION

Powers of Polynomials

To quickly find the power of a polynomial we can use the binomial theorem. The first three powers of the binomial $(x+a)$ are given below.

1. $(x+a)^2 = x^2 + 2ax + a^2$
2. $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$
3. $(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$

VIDEO EXAMPLE 1.8 - POLYNOMIAL POWERS

Factoring Techniques

Procedures for factoring are illustrated below.

1. Look for a common factor.
2. To factor a quadratic of the form $x^2 + bx + c$, look for two real numbers p and q satisfying $p + q = b$ and $pq = c$.
3. To factor a quadratic of the form $ax^2 + bx + c$ where $a \neq 0$ and $a \neq 1$ quickly, use the guess and check method illustrated in the video.
4. To factor a difference of squares, use conjugate pairs: $x^2 - p^2 = (x + p)(x - p)$. **A sum of two squares does not factor over the real numbers.**
5. To factor a sum or difference of cubes, use the factoring formulas below. **The trinomial you obtain by factoring a sum or difference of cubes does not factor further.**
   i. $x^3 + p^3 = (x + p)(x^2 - px + p^2)$
   ii. $x^3 - p^3 = (x - p)(x^2 + px + p^2)$
6. More techniques for factoring general polynomial functions are given in Chapter 3.

VIDEO EXAMPLE 1.9 - COMMON FACTORING

VIDEO EXAMPLE 1.10 - FACTORING QUADRATICS 1

VIDEO EXAMPLE 1.11 - FACTORING QUADRATICS 2

VIDEO EXAMPLE 1.12 - FACTORING DIFFERENCE OF SQUARES

VIDEO EXAMPLE 1.13 - FACTORING SUM/DIFFERENCE OF CUBES

Rational Expressions: Addition/Subtraction

A function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x) \neq 0$ are polynomials is called a rational function. More on rational functions is given in Chapter 3. For now, we will refer to them as rational expressions.

When adding or subtracting two rational expressions, use the steps below.

1. Fully factor all denominators.
2. Determine the common denominator consisting of the fewest factors.
3. When all rational expressions have the same denominator, add/subtract the numerators while keeping the common denominator.
4. **Leave your denominator in fully factored form.**
5. Check your numerator to see if it factors. Cancel any common factors and note any restrictions.

VIDEO EXAMPLE 1.14 - ADDING/SUBTRACTING RATIONAL EXPRESSIONS 1

VIDEO EXAMPLE 1.15 - ADDING/SUBTRACTING RATIONAL EXPRESSIONS 2

Rational Expressions: Multiplication/Division

When multiplying two rational expressions, use the steps below.

1. Fully factor all numerators and denominators.
2. Cancel any common factors and note any restrictions.
3. Multiply the numerators and denominators.
4. **Leave your solution in fully factored form.**

VIDEO EXAMPLE 1.16 - MULTIPLYING RATIONAL EXPRESSIONS

When dividing two rational expressions, use the steps below.

1. Fully factor all numerators and denominators.
2. Multiply the first rational expression by the reciprocal of the second rational expression.
3. Cancel any common factors and note any restrictions.

4. **Leave your solution in fully factored form.**

VIDEO EXAMPLE 1.17 - DIVIDING RATIONAL EXPRESSIONS

Simplifying Rational Expressions

The video below gives more examples of simplifying difficult rational expressions.

VIDEO EXAMPLE 1.18 - SIMPLIFYING RATIONAL EXPRESSIONS

Rationalization

When we rationalize a denominator, we are trying to find an equivalent expression without a square root in the denominator. To rationalize the denominator, first identify the form from the chart below, then multiply by the given rational expression. A, B and C represent any non-zero expressions.

Form	Multiply by
$\frac{A}{\sqrt{B}}$	$\frac{\sqrt{B}}{\sqrt{B}}$
$\frac{A}{\sqrt{B} \pm \sqrt{C}}$	$\frac{\sqrt{B} \mp \sqrt{C}}{\sqrt{B} \mp \sqrt{C}}$
$\frac{A}{\sqrt{B} \pm C}$	$\frac{\sqrt{B} \mp C}{\sqrt{B} \mp C}$

VIDEO EXAMPLE 1.19 - RATIONALIZING THE DENOMINATOR

When we rationalize a numerator, we are trying to find an equivalent expression without a square root in the numerator.To rationalize the numerator, first identify the form from the chart below, then multiply by the given rational expression. A, B and C represent any non-zero expressions.

Form	Multiply by
$\frac{\sqrt{A}}{B}$	$\frac{\sqrt{A}}{\sqrt{A}}$
$\frac{\sqrt{B} \pm \sqrt{C}}{A}$	$\frac{\sqrt{B} \mp \sqrt{C}}{\sqrt{B} \mp \sqrt{C}}$
$\frac{\sqrt{B} \pm C}{A}$	$\frac{\sqrt{B} \mp C}{\sqrt{B} \mp C}$

VIDEO EXAMPLE 1.20 - RATIONALIZING THE NUMERATOR