Precalculus Workshop - Logarithms

The Logarithmic Operation

Let a and b be positive real numbers. If $a > 0$ and $a \neq 1$, we define the logarithm of b with base a by

$$\log_a(b) = x \iff a^x = b.$$

That is, $\log_a(b)$ is the solution of the equation $a^x = b$.

Thus, the logarithmic function $y = \log_a(x)$, $(x > 0)$ is the inverse function of $f(x) = a^x$ (more in inverse functions). The logarithmic and exponential operations are therefore *inverse operations* and satisfy the following properties of inverse functions.

1. $\log_a(a^x) = x \quad (x \in \mathbb{R})$
2. $a^{\log_a(x)} = x \quad (x > 0)$

VIDEO EXAMPLE 4.1 - SIMPLIFYING LOGARITHMIC EXPRESSIONS

Precalculus Workshop - Logarithms

Laws of Logarithms

The following list of important logarithm laws **should be memorized.** Let $a > 0$ and $a \neq 1$. Suppose that $p > 0$, $q > 0$ and k are real numbers. Then we have the following:

1. $\log_a(pq) = \log_a(p) + \log_a(q)$
2. $\log_a\left(\frac{p}{q}\right) = \log_a(p) - \log_a(q)$
3. $\log_a(p^k) = k\log_a(p)$.

If you are taking calculus, these laws will be discussed in greater detail in this class.

VIDEO EXAMPLE 4.2 - MULTIPLICATION LAW OF LOGS

VIDEO EXAMPLE 4.3 - DIVISION LAW OF LOGS

VIDEO EXAMPLE 4.4 - POWER LAW OF LOGS

VIDEO EXAMPLE 4.5 - COMBINING LOG LAWS

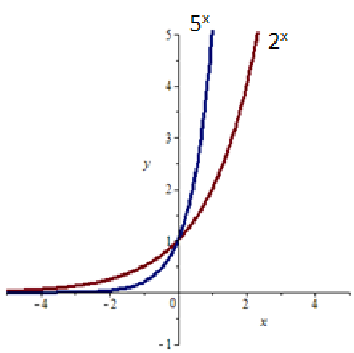
Precalculus Workshop - Logarithms

Exponential Functions

An exponential function has the form $f(x) = a^x$, where $a > 0$, $a \neq 1$. Any exponential function of this form has four traits you should know.

1. An exponential function passes through the point $(0, 1)$, that is $a^0 = 1$.
2. An exponential function has a horizontal asymptote at $y = 0$.
3. The domain of an exponential function is $(-\infty, \infty) = \{x \mid x \in \mathbb{R}\}$.
4. The range of an exponential function is $(0, \infty) = \{y \mid y > 0\}$.

The exponential functions $f(x) = 2^x$ and $g(x) = 5^x$ have been plotted below for your reference.



The Natural Exponential Function

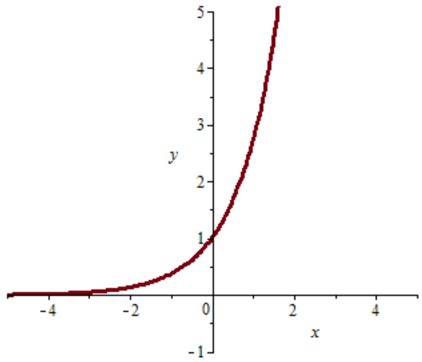
Euler's constant, denoted by the letter e, is defined as the irrational number satisfying the following limit (you will learn about limits in your calculus course):

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \mathrm{e} \approx 2.71828\ldots.$$

Using Euler's constant, we can define an exponential function $f(x) = \mathrm{e}^x$ called the *natural exponential function*. The function $f(x) = \mathrm{e}^x$ satisfies all four traits of an exponential function.

1. The natural exponential function passes through the point $(0, 1)$, that is $\mathrm{e}^0 = 1$.
2. The natural exponential function has a horizontal asymptote at $y = 0$.
3. The domain of the natural exponential function is $(-\infty, \infty) = \{x \mid x \in \mathbb{R}\}$.
4. The range of the natural exponential function is $(0, \infty) = \{y \mid y > 0\}$.

The graph of $f(x) = \mathrm{e}^x$ is given below for your reference.



Algebraically, the natural exponential function e^x satisfies all power rules.

1. $\mathrm{e}^0 = 1$
2. $\mathrm{e}^1 = \mathrm{e}$
3. $\mathrm{e}^{-m} = \dfrac{1}{\mathrm{e}^m}$
4. $\mathrm{e}^m \mathrm{e}^n = \mathrm{e}^{m+n}$
5. $\dfrac{\mathrm{e}^m}{\mathrm{e}^n} = \mathrm{e}^{m-n}$
6. $(\mathrm{e}^m)^n = \mathrm{e}^{mn}$

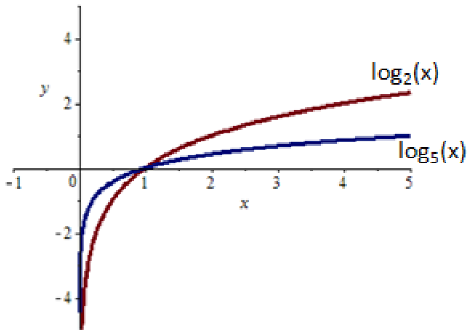
VIDEO EXAMPLE 4.6 - SIMPLIFYING NATURAL EXPONENTIAL EXPRESSIONS

Logarithmic Functions

A logarithmic function has the form $f(x) = \log_a(x)$, where $a > 1$. Any logarithmic function of this form has four traits you should know.

1. A logarithmic function passes through the point $(1, 0)$, that is $\log_a(1) = 0$.
2. A logarithmic function has a vertical asymptote at $x = 0$.
3. The domain of a logarithmic function is $(0, \infty) = \{x \mid x > 0\}$.
4. The range of a logarithmic function is $(-\infty, \infty) = \{y \mid y \in \mathbb{R}\}$.

The logarithmic functions $f(x) = \log_2(x)$ and $g(x) = \log_5(x)$ have been plotted below for your reference.



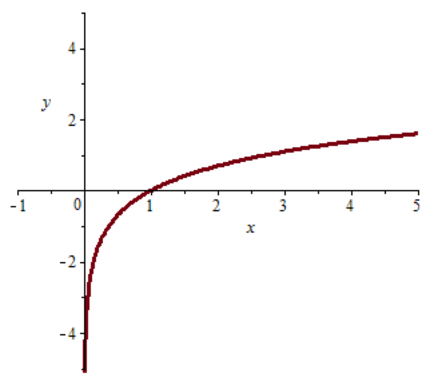
THE NATURAL LOGARITHMIC FUNCTION

Since e is a positive real number not equal to one, we can define the logarithmic function $f(x) = \log_{\mathrm{e}}(x) = \ln(x)$, called the *natural logarithmic function.*

The function $f(x) = \ln(x)$ satisfies all four traits of a logarithmic function.

1. The natural logarithmic function passes through the point $(1, 0)$, that is $\ln(1) = 0$.
2. The natural logarithmic function has a vertical asymptote at $x = 0$.
3. The domain of the natural logarithmic function is $(0, \infty) = \{x \mid x > 0\}$.
4. The range of the natural logarithmic function is $(-\infty, \infty) = \{y \mid y \in \mathbb{R}\}$.

The graph of $f(x) = \ln(x)$ is given below for your reference.



The natural logarithmic function $\ln(x)$ satisfies all logarithm laws.

1. $\ln(pq) = \ln(p) + \ln(q)$
2. $\ln\left(\dfrac{p}{q}\right) = \ln(p) - \ln(q)$
3. $\ln(p^k) = k\ln(p)$

Finally, the natural logarithm and the natural exponential are inverse functions of each other. This leads to the following important identities:

1. $\ln(e^x) = x$, $x \in \mathbb{R}$
2. $e^{\ln(x)} = x$, $x > 0$
3. $\ln(1) = 0$
4. $\ln(e) = 1$

VIDEO EXAMPLE 4.7 - SIMPLIFYING NATURAL LOGARITHMIC EXPRESSIONS 1

VIDEO EXAMPLE 4.8 - SIMPLIFYING NATURAL LOGARITHMIC EXPRESSIONS 2

Precalculus Workshop - Logarithms

Exponential Equations

To solve for x in an equation of the form $a^x = b$, we can apply the logarithm function with base a to both sides of the equation.

1. If $a^x = b$, then $\log_a(a^x) = \log_a(b)$. This simplifies to $x = \log_a(b)$ since the logarithmic and exponential functions with base a are inverses of each other.

VIDEO EXAMPLE 4.9 - EXPONENTIAL EQUATIONS I

VIDEO EXAMPLE 4.10 - EXPONENTIAL EQUATIONS II

VIDEO EXAMPLE 4.11 - EXPONENTIAL EQUATIONS III

Logarithmic Equations

To solve for x in an equation of the form $\log_a(x) = b$, we can apply the exponential function with base a to both sides of the equation.

1. If $\log_a(x) = b$, then $a^{\log_a(x)} = a^b$. This simplifies to $x = a^b$ since the logarithmic and exponential functions with base a are inverses of each other.

VIDEO EXAMPLE 4.12 - LOGARITHMIC EQUATIONS I

VIDEO EXAMPLE 4.13 - LOGARITHMIC EQUATIONS II