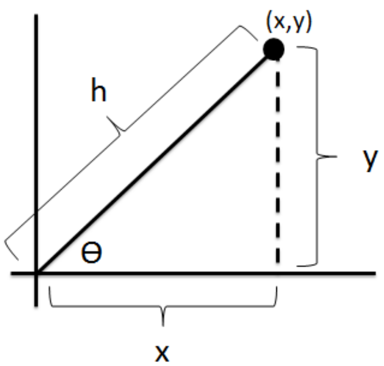
Precalculus Workshop - Trigonometry

Ratios of Right-Angled Triangles

Given a point in the xy-plane, we can create a right-angled triangle by considering the diagram below. The angle θ is called a reference angle. If we compare any two side lengths of this triangle, we can form six different ratios. These ratios are called trigonometric ratios and are defined below.



$$\cos(\theta) = \frac{x}{h} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{h}{x}$$

$$\sin(\theta) = \frac{y}{h} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{h}{y}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{y}{x} \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{x}{y}$$

VIDEO EXAMPLE 5.1 - DETERMINING TRIGONOMETRIC RATIOS 1

The CAST Rule

Recall the CAST Rule, which tells us which trigonometric ratios are exclusively positive in each quadrant.

1. $\cos(\theta)$ is exclusively positive in Q4, the fourth quadrant of the xy-plane.
2. All ratios are positive in Q1.
3. $\sin(\theta)$ is exclusively positive in Q2.
4. $\tan(\theta)$ is exclusively positive in Q3.

The CAST Rule can be visualized with the following diagram:

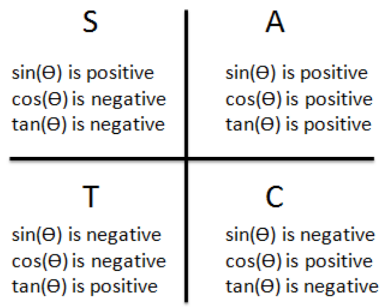
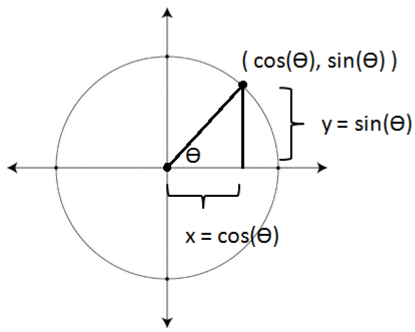


Figure 1: Visualization of the CAST Rule.

VIDEO EXAMPLE 5.2 - DETERMINING TRIGONOMETRIC RATIOS 2

THE UNIT CIRCLE

It turns out that a nice value to take for the hypotenuse is $h = 1$. When this happens, we get points in the xy-plane on the unit circle. Notice that sine and cosine become



$$\cos(\theta) = \frac{x}{h} = x$$

$$\sin(\theta) = \frac{y}{h} = y$$

which means $\cos(\theta)$ is the x-coordinate of the point on the unit circle at angle θ, and $\sin(\theta)$ is the y-coordinate of the point on the unit circle at angle θ, where θ represents the angle measured from the positive x-axis in the counter-clockwise direction. All of our trigonometric ratios can be thought of as real-valued trigonometric functions of θ.

You should memorize the values of $\sin(\theta)$ and $\cos(\theta)$ at the important angles in the first quadrant of the unit circle:

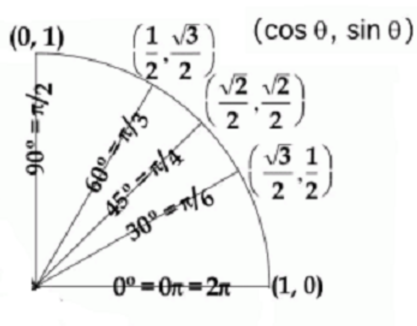


Figure 1: Special angles in the first quadrant of the unit circle.

The video examples below will show you how to use the CAST Rule to evaluate trigonometric functions at important angles in the other quadrants.

VIDEO EXAMPLE 5.3 - A WAY TO REMEMBER THE UNIT CIRCLE

VIDEO EXAMPLE 5.4 - EVALUATING TRIG FUNCTIONS 1

VIDEO EXAMPLE 5.5 - EVALUATING TRIG FUNCTIONS 2

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Basic Trigonometric Graphs

Recall that $\cos(\theta)$ and $\sin(\theta)$ are 2π-periodic. This means they repeat every 2π radians. It is recommended that you know the graphs of $\cos(\theta)$ and $\sin(\theta)$ on the interval $[0, 2\pi]$. The graphs of these functions, along with five important points are given below.

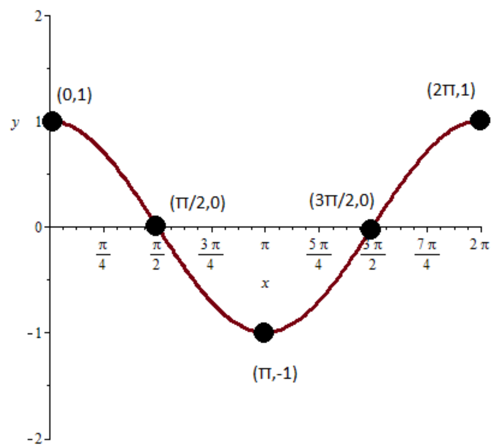

Figure 1: Plot of $\cos(\theta)$ on the interval $[0, 2\pi]$.

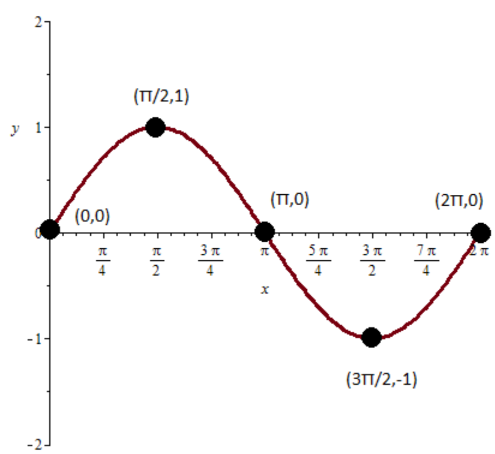

Figure 2: Plot of $\sin(\theta)$ on the interval $[0, 2\pi]$.

Recall that $\tan(\theta)$ is π-periodic. This means tangent repeats every π radians. Note that since $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, $\tan(\theta)$ is undefined when $\cos(\theta) = 0$. Also, $\tan(\theta) = 0$ whenever $\sin(\theta) = 0$. The graph of the tangent function is given below.

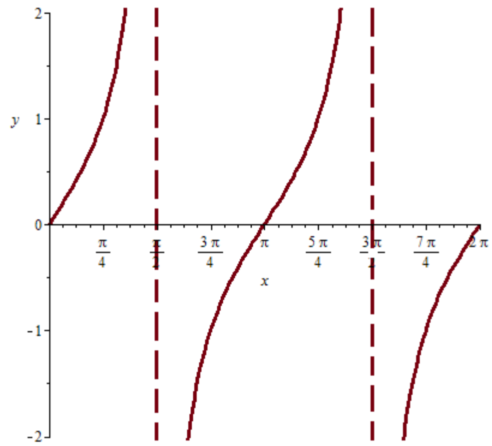

Figure 3: Plot of $\tan(\theta)$ on the interval $[0, 2\pi]$.

Shifts of Cosine and Sine

Below is a chart summarizing transformations of the function $f(x)$.

Form	Description
$f(x) + C$	Shift up ($C > 0$) or down ($C < 0$) by C units
$f(x + C)$	Shift left ($C > 0$) or right ($C < 0$) by C units
$-f(x)$	Reflection over the x-axis
$f(-x)$	Reflection over the y-axis
$Cf(x)$	Stretch ($C > 1$) or compression ($0 < C < 1$) by a factor of C in the y-axis
$f(Cx)$	Stretch ($0 < C < 1$) or compression ($C > 1$) by a factor of C in the x-axis

VIDEO EXAMPLE 5.6 - COSINE TRANSFORMATION

VIDEO EXAMPLE 5.7 - SINE TRANSFORMATION

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Pythagorean Identities

Provided that all trigonometric functions below are defined for x, the three pythagorean identities are:

1. $\sin^2(x) + \cos^2(x) = 1$
2. $\tan^2(x) + 1 = \sec^2(x)$
3. $1 + \cot^2(x) = \csc^2(x)$.

VIDEO EXAMPLE 5.8 - PYTHAGOREAN IDENTITIES

Sum and Difference Formulas

Recall the sum and difference formulas for sine and cosine:

1. $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
2. $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
3. $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
4. $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$.

VIDEO EXAMPLE 5.9 - SUM AND DIFFERENCE FORMULAS

Double Angle Identities

If we substitute $y = x$ into the addition formulas above, we obtain the double angle formulas for sine and cosine. Since $\sin^2(x) + \cos^2(x) = 1$, there are three alternate forms for the cosine double-angle identity.

1. $\sin(2x) = 2\sin(x)\cos(x)$
2. $\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$

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Basic Trigonometric Equations

We will begin by solving trigonometric equations of the form

$$\begin{aligned}\cos(x) &= a\\ \sin(x) &= a\\ \tan(x) &= a.\end{aligned}$$

To solve these types of trigonometric equations, we can follow the procedure below.

1. Solve the equation in quadrant one using the reference angle. You will have to consider $|a|$ if $a < 0$.
2. Use the CAST Rule to determine if there are any other quadrants you need to consider. Solve the equation in these quadrants as well.
3. If you have a specific interval for x, ensure all values of x fall within this interval.
4. If you do not have a specific interval for x, you must use a general solution. For $\sin(x)$ and $\cos(x)$, we add "$+2k\pi$" to each solution. For $\tan(x)$ we add "$+k\pi$" to each solution.

VIDEO EXAMPLE 5.10 - BASIC TRIGONOMETRIC EQUATIONS 1

VIDEO EXAMPLE 5.11 - BASIC TRIGONOMETRIC EQUATIONS 2

Reciprocal Trigonometric Equations

We will next see how to solve trigonometric equations of the form

$$\begin{aligned}\sec(x) &= a\\ \csc(x) &= a\\ \cot(x) &= a.\end{aligned}$$

To solve these types of trigonometric equations, we can follow the procedure below.

1. Change the equation into a basic trigonometric equation using one of the reciprocal identities:

$$\begin{aligned}\sec(x) &= \frac{1}{\cos(x)}\\ \csc(x) &= \frac{1}{\sin(x)}\\ \cot(x) &= \frac{1}{\tan(x)}.\end{aligned}$$

2. Solve the new equation using the method described above.

VIDEO EXAMPLE 5.12 - RECIPROCAL TRIGONOMETRIC EQUATIONS 1

Equations Involving Multiple Angles

A multiple angle equation is one involving a multiple of the variable within the trigonometric function (like $2x$ or $3x$). To solve these types of trigonometric equations, we can follow the procedure below.

1. If the angle of your trigonometric function is bx, make a substitution $\alpha = bx$.
2. Now you should have either a basic trigonometric equation, or a reciprocal trigonometric equation.
3. Solve this equation for α using the methods outlined above. **Ensure that you form the most general solution.**
4. Replace α with bx and divide all solutions by b.
5. If you do not have a specific interval for x, you now have your most general solution and you are done.
6. If you do have an interval for x, substitute values of k into your general solution to determine the specific values of x that lie in the given interval.

VIDEO EXAMPLE 5.13 - MULTIPLE ANGLE EQUATIONS 1

VIDEO EXAMPLE 5.14 - MULTIPLE ANGLE EQUATIONS 2

Other Techniques

The videos below outline some other common techniqeus to be aware of when solving trigonometric equations.

VIDEO EXAMPLE 5.15 - FACTORING

VIDEO EXAMPLE 5.16 - USING AN IDENTITY